

CERTAIN WEIGHTED AVERAGES OF GENERALIZED RAMANUJAN SUMS

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ABSTRACT. We derive certain identities involving various known arithmetical functions and a generalized version of Ramanujan sum. L. Tóth constructed certain weighted averages of Ramanujan sums with various arithmetic functions as weights. We choose a generalization of Ramanujan sum given by E. Cohen and derive the weighted averages corresponding to the versions of the weighted averages established by Tóth.

KEYWORDS: Generalized Ramanujan sum, weighted averages, Jordan totient function, Gamma function, Bernoulli numbers, Holder evaluation

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1. INTRODUCTION

Ramanujan sum is a certain sum of powers of roots of unity defined and used seriously for the first time in the literature by Srinivasa Ramanujan. His paper *On certain trigonometric sums* published in 1918 ([15]) discusses various infinite series expressions for many well known arithmetical functions in the form of a power series. His interest in the sums originated in his desire to “obtain expressions for a variety of well-known arithmetical functions of n in the form of a series $\sum_s a_s c_s(n)$ ”. There have been a lot of analysis dealing with this particular aspect of Ramanujan sums (See, for example, [13]). On the other hand, it found immense use in classical character theory. It was proved that the sums can be used to establish the integrality of the character values for the symmetric group ([12]). However, perhaps the most famous appearance of Ramanujan sums is their crucial role in Vinogradov’s proof (See [14]) that every sufficiently large odd number is the sum of three primes.

In more recent years, Ramanujan sums have appeared in various other seemingly unrelated problems. Being interested in the applications of the Ramanujan sums, many mathematicians later tried to generalize it to find more and more applications. One of the most popular generalization was given by E. Cohen. After that, various other generalizations were discussed in many papers including that of C. S. Venkataraman and R. Sivaramakrishnan [19] and J. Chidambaraswamy ([5]).

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In this paper we are interestd in establishing certain weighted averages of generalized Ramanujan sums. The generalization we choose was given by E. Cohen. We remark that our identities are based on similar kind of averages derived by L. Tóth in [18] using the classical Ramanujan sums. In between, we derive another identity using the Jordan totient function which could be of independent interest.

2. NOTATIONS

Most of the notations, functions, and identities we introduce below are standard and can be found in [4] or [9]. We use the following standard notations: \mathbb{N} the set of all natural numbers, \mathbb{Z} the set of all integers, \mathbb{R} the set of all real numbers and \mathbb{C} the set of all complex numbers. μ will denote the Mobius function on \mathbb{N} . Note that for any given $k \in \mathbb{N}$, we have

$$\sum_{d|k} \mu(d) = 0$$

Let $[x]$ denote the integer part of x , and $\tau(n)$ and $\sigma(n)$ denote the number and sum of divisors of n respectively. For two arithmetical functions f and g , $f * g$ denote their Dirichlet convolution (Dirichlet product). Note that this product is commutative. The arithmetic function N is defined as, for $n \in \mathbb{N}$,

$$N^\alpha(n) := n^\alpha$$

for any $\alpha \in \mathbb{R}$. For $\alpha = 1$, we will simply write $N(n)$ instead of writing $N^1(n)$, and $N^0(n) := 1$ for all n .

The von Mangoldt function over \mathbb{N} is denoted by $\Lambda(n)$. The Euler totient function is denoted by φ and it satisfies the identity

$$\varphi(n) = n \prod_{\substack{p|n \\ p \text{ prime}}} \left(1 - \frac{1}{p}\right)$$

A well known generalization of this identity gives the Jordan totient function of order $s \in \mathbb{N}$:

$$J_s(n) := n^s \prod_{\substack{p|n \\ p \text{ prime}}} \left(1 - \frac{1}{p^s}\right)$$

where $n \in \mathbb{N}$.

We would be using the following identity for the Jordan totient function:

Proposition 1. ([4], Chapter 2, Exer. 17) For $n \geq 1$, we have

$$J_s(n) = \sum_{d|n} d^s \mu(n/d) = \sum_{d|n} (n/d)^s \mu(d)$$

For $x \in \mathbb{R}, x > 0$, the Gamma function Γ is defined by

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt$$

Now we recall the definition of the Bernoulli Polynomials and Bernoulli numbers. For any $x \in \mathbb{C}$ define the functions $B_n(x)$ by the equation

$$\frac{ze^{xz}}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n(x)}{n!} z^n \text{ where } |z| < 2\pi$$

If we put $x = 0$, we get (with the convention that $B_n := B_n(0)$), then

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n \text{ where } |z| < 2\pi$$

$B_n(x)$ are called Bernoulli polynomials and $B_n = B_n(0)$ are called Bernoulli numbers.

Following are some important properties of Bernoulli polynomials (See section 12.2 of [4]) which we use later:

Proposition 2. *For $n \in \mathbb{N}, n \geq 2$, the Bernoulli numbers satisfy*

- (1) $B_n(0) = B_n(1)$, and
- (2) $B_n = \sum_{k=0}^n \binom{n}{k} B_k$

Next we define the generalized gcd function:

Definition 3. *The generalized GCD function on $\mathbb{N} \times \mathbb{N}$ denoted by $(j, k^s)_s$ is defined to give the largest $d \in \mathbb{N}$ such that $d|k$ and $d^s|j$. Therefore, $(j, k^s)_s = 1$ means that no natural number $d \geq 1$ exists such that $d|k$ and $d^s|j$.*

When $s = 1$, the generalized GCD function becomes the usual GCD function.

For $k \in \mathbb{N}, j \in \mathbb{Z}$, the Ramanujan sum defined as the sum

$$c_k(j) = \sum_{\substack{m=1 \\ \gcd(m,k)=1}}^k e^{2\pi i j m / k}$$

In [6], E. Cohen generalized this to

Definition 4. *For $s, k \in \mathbb{N}, j \in \mathbb{Z}$, the generalized Ramanujan sum is defined as*

$$c_k^{(s)}(j) = \sum_{\substack{m=1 \\ (j, k^s)_s=1}}^k e^{2\pi i j m / k^s}$$

Recall that, for Ramanujan sum, we have

$$c_k(j) = \sum_{d|(k,j)} d \mu\left(\frac{k}{d}\right) = \sum_{d|(k,j)} \frac{k}{d} \mu(d)$$

The following similar identity holds for the generalized Ramanujan sum ([6] Section 2):

Proposition 5.

$$(1) \quad c_k^{(s)}(j) = \sum_{d|(j, k^s)_s} d^s \mu\left(\frac{k}{d}\right) = \sum_{\substack{d|j^s \\ d|k}} d^s \mu\left(\frac{k}{d}\right)$$

Holder's (See [11]) evaluation of the Ramanujan sums is given by

$$c_k(j) = \frac{\varphi(k)\mu(k/\gcd(k, j))}{\varphi(k/\gcd(k, j))} \text{ where } k \in \mathbb{N}, j \in \mathbb{Z}$$

For the generalized sum, Cohen showed that ([8], section 2)

Proposition 6. (*Generalized Holder's identity*)

$$(2) \quad c_k^{(s)}(j) = \frac{J_s(k)\mu(k/(j, k^s)_s)}{J_s(k/(j, k^s)_s)} \text{ where } j, k, s \in \mathbb{N}$$

Definition 7. For $k, n \in \mathbb{N}$, the function $\theta_k(n)$ is defined as

$$\theta_k(n) := \begin{cases} 1, & \text{if } (k, n) = 1 \\ 0, & \text{if } (k, n) > 1 \end{cases}$$

Let \mathcal{F} denote the \mathbb{C} -linear space of all arithmetic functions with the usual operations.

Definition 8. An arithmetic function $f \in \mathcal{F}$ is said to be k -periodic for a fixed $k \in \mathbb{N}$ if $f(n+k) = f(k)$ for all $n \in \mathbb{N}$.

The following proposition which we use later, can be found, for example, in [16].

Proposition 9. For $k \in \mathbb{N}$, every k -periodic $f \in \mathcal{F}$ has a Fourier expansion of the form

$$f(k) = \sum_{j=1}^k g(j) \exp(2\pi i j n / k)$$

where the Fourier coefficients $g(j)$ are unique, and are in fact given by the equation

$$g(m) = \frac{1}{k} \sum_{j=1}^k f(k) \exp(-2\pi i j n / k)$$

Any other notations and definitions we use will be introduced later as and when required.

3. PROBLEMS

For $r \in \mathbb{N}$, E. Alkan established an identity for the weighted average

$$\frac{1}{k^{r+1}} \sum_{j=1}^k j^r c_k(j)$$

based on which he could establish many results in [1] and [2]. If we denote the above weighted average by $S_r(k)$, the identity given by E. Alkan is precisely the following:

Proposition 10.

$$(3) \quad S_r(k) = \frac{\varphi(k)}{2k} + \frac{1}{r+1} \sum_{m=0}^{\lfloor r/2 \rfloor} \binom{r+1}{2m} B_{2m} \prod_{p|k} \left(1 - \frac{1}{p^{2m}}\right)$$

This identity appeared as Eq. 2.19 in [2]. He used this identity to prove exact formulas for certain mean square averages of special values of L -functions. Alkan described these mean square averages in [1] and proved an asymptotic formula for $\sum_{k \leq x} S_r k$ in [2] based on the identity given above.

How did he proceed to establish the identity is, that, he used Holder's evaluation of the Ramanujan sums [11] and applied the formula

$$(4) \quad \sum_{\substack{j=1 \\ \gcd(j,n)=1}}^n j^r = \frac{n^{r+1}}{r+1} \sum_{m=0}^{\lfloor r/2 \rfloor} \binom{r+1}{2m} \frac{B_{2m}}{n^{2m}} \prod_{p|n} (1 - p^{2m-1}) \quad (n, r \in \mathbb{N}, n > 1)$$

(See [17] for details) and then considered the cases r even and r odd separately. The identity (3) and its proof appears in [3] also.

The proof of the identity (3) given in [1] was a little bit complicated. A simpler proof this identity (3) appeared in a paper [18] of L.Tóth. In addition, in the same paper, Tóth derived identities for some other weighted averages of the Ramanujan sums with weights concerning logarithms, values of arithmetic functions for gcd's, the Gamma function, the Bernoulli numbers, binomial coefficients, and the exponential function. Our focus will be on establishing the weighted averages of Tóth for the generalized Ramanujan sums (of E. Cohen) replacing the classical Ramanujan sums. Considering that the generalized Ramanujan sum with $s = 1$ becomes the classical Ramanujan sum, it may not be very surprising to note that, the identities we establish here will resemble Tóth's identities when we put $s = 1$. In fact, the way in which we proceed to establish the identities closely resembles the steps given in [18].

4. ALKAN'S IDENTITY

As we mentioned earlier, the most important computations in [1] and [2] of E. Alkan depends heavily on the identity given in Proposition (10). The identity in its final form consists of the Euler totient function and the Jordan totient function. We propose and prove a similar identity for the generalized Ramanujan sum consisting of similar type of functions. Our identity is precisely the following:

Proposition 11. *For $s, k, r \in \mathbb{N}$, define the weighted average $S_r^{(s)}(k)$ by*

$$(5) \quad S_r^{(s)}(k) = \frac{1}{k^{s(r+1)}} \sum_{j=1}^{k^s} j^r c_k^{(s)}(j)$$

Then,

$$(6) \quad S_r^{(s)}(k) = \frac{J_s(k)}{2k} + \frac{1}{r+1} \sum_{m=0}^{\lfloor r/2 \rfloor} \binom{r+1}{2m} B_{2m} \frac{J_s(k)}{k^{2ms}}$$

Proof. Using identity (1), we get

$$\begin{aligned}
S_r^{(s)}(k) &= \frac{1}{k^{s(r+1)}} \sum_{j=1}^{k^s} j^r \sum_{\substack{d|j^s \\ d|k}} d^s \mu\left(\frac{k}{d}\right) \\
&= \frac{1}{k^{s(r+1)}} \sum_{d|k} d^s \mu\left(\frac{k}{d}\right) \sum_{n=1}^{k^s/d^s} (nd^s)^r \\
&= \sum_{d|k} \frac{d^{s(r+1)}}{k^{s(r+1)}} \mu\left(\frac{k}{d}\right) \sum_{n=1}^{k^s/d^s} n^r
\end{aligned}$$

Now from [9], Prop. 9.2.12, for every $N, r \in \mathbb{N}$, we have

$$\sum_{n=1}^N n^r = \frac{1}{r+1} \left(N^{r+1} + \frac{r+1}{2} N^r + \sum_{m=2}^r \binom{r+1}{m} B_m N^{r+1-m} \right)$$

Note that $B_0 = 1$, and $B_1 = -\frac{1}{2}$ (See [4], section 12.12). Also, for $m \geq 1$, $B_{2m+1} = 0$ (See [4] Theorem 12.16). Therefore, B_3, B_5, \dots are all equal to 0. So we can rewrite the above expression as

$$\begin{aligned}
\sum_{n=1}^N n^r &= \frac{1}{r+1} \left((-1)^0 B_0 \binom{r+1}{0} N^{r+1-0} + (-1)^1 B_1 \binom{r+1}{1} N^{r+1-1} \right. \\
&\quad \left. + \sum_{m=2}^r \binom{r+1}{m} B_m N^{r+1-m} \right)
\end{aligned}$$

In this expression, after B_1 , only even order Bernoulli numbers survive and so we need to consider only such terms in the summation. Rewriting using this information

$$\begin{aligned}
\sum_{n=1}^N n^r &= \frac{1}{r+1} (-1)^1 B_1 \binom{r+1}{1} N^{r+1-1} + \frac{1}{r+1} \sum_{m=0}^{\lfloor \frac{r}{2} \rfloor} \binom{r+1}{2m} B_{2m} N^{r+1-2m} \\
&= \frac{N^r}{2} + \frac{1}{r+1} \sum_{m=0}^{\lfloor \frac{r}{2} \rfloor} \binom{r+1}{2m} B_{2m} N^{r+1-2m}
\end{aligned}$$

Using this new expression, we get

$$\begin{aligned}
S_r^{(s)}(k) &= \sum_{d|k} \frac{d^{s(r+1)}}{k^{s(r+1)}} \mu\left(\frac{k}{d}\right) \left[\left(\frac{k^s}{d^s}\right)^r \frac{1}{2} + \frac{1}{r+1} \sum_{m=0}^{\lfloor \frac{r}{2} \rfloor} \binom{r+1}{2m} B_{2m} \left(\frac{k^s}{d^s}\right)^{r+1-2m} \right] \\
&= \frac{1}{2} \sum_{d|k} \frac{d^s}{k^s} \mu\left(\frac{k}{d}\right) + \frac{1}{r+1} \sum_{m=0}^{\lfloor \frac{r}{2} \rfloor} \binom{r+1}{2m} B_{2m} \sum_{d|k} \frac{d^{2ms}}{k^{2ms}} \mu\left(\frac{k}{d}\right) \\
&= \frac{1}{2} \frac{J_s(k)}{k^s} + \frac{1}{r+1} \sum_{m=0}^{\lfloor \frac{r}{2} \rfloor} \binom{r+1}{2m} B_{2m} \frac{J_{2ms}(k)}{k^{2ms}}
\end{aligned}$$

and this is what we claimed. \square

When $s = 1$, the above identity becomes the one given by Alkan (Prop. 10).

5. log AS WEIGHT

We proceed now to generalize the other weighted averages computed by Tóth([18]). The next one uses the logarithmic function as weight.

Proposition 12. *For every $s, k \in \mathbb{N}$, we have*

$$\frac{1}{k} \sum_{j=1}^k \log j c_k^{(s)}(j) = s\Lambda(k) + \sum_{d|k} \frac{d^s}{k} \mu\left(\frac{k}{d}\right) \log\left(\frac{k}{d^s}\right)!$$

Proof. We use identity (1) to proceed:

$$\begin{aligned} \sum_{j=1}^k \log j c_k^{(s)}(j) &= \sum_{j=1}^k \log j \sum_{\substack{d|j^s \\ d|k}} d^s \mu\left(\frac{k}{d}\right) \\ &= \sum_{d|k} d^s \mu\left(\frac{k}{d}\right) \sum_{m=1}^{\frac{k}{d^s}} \log(md^s) \\ &= \sum_{d|k} d^s \mu\left(\frac{k}{d}\right) \sum_{m=1}^{\frac{k}{d^s}} [\log(1 \cdot d^s) + \log(2 \cdot d^s) + \dots + \log\left(\frac{k}{d^s} d^s\right)] \\ &= \sum_{d|k} d^s \mu\left(\frac{k}{d}\right) \log\left[\left(\frac{k}{d^s}\right)! (d^s)^{\frac{k}{d^s}}\right] \\ &= \sum_{d|k} d^s s \frac{k}{d^s} \log d \mu\left(\frac{k}{d}\right) + \sum_{d|k} d^s \mu\left(\frac{k}{d}\right) \log\left(\frac{k}{d^s}\right)! \\ &= sk \sum_{d|k} \mu\left(\frac{k}{d}\right) \log d + \sum_{d|k} d^s \mu\left(\frac{k}{d}\right) \log\left(\frac{k}{d^s}\right)! \end{aligned}$$

Note that $\sum_{d|k} \mu\left(\frac{k}{d}\right) \log d = (\mu * \log)(k) = \Lambda(k)$. So we get

$$\frac{1}{k} \sum_{j=1}^k \log j c_k^{(s)}(j) = s\Lambda(k) + \sum_{d|k} \frac{d^s}{k} \mu\left(\frac{k}{d}\right) \log\left(\frac{k}{d^s}\right)!$$

□

6. FUNCTION OF GENERALIZED GCD AS WEIGHT

Now we proceed to establish an identity with weight as the generalized GCD function. But instead, we prove it generally for arithmetic functions composed with the GCD function. This, on taking the first function to be the function N as a special case, we get the weight as the generalized GCD function.

Proposition 13. For $s, k, j \in \mathbb{N}$

$$\sum_{j=1}^{k^s} f((j, k^s)_s) c_k^{(s)}(j) = J_s(k) [(f \circ N^s) * (\mu \circ N^s)](k)$$

Proof. We use the generalized Holder's evaluation (2) to start off.

$$\sum_{j=1}^{k^s} f((j, k^s)_s) c_k^{(s)}(j) = \sum_{j=1}^{k^s} f((j, k^s)_s) \frac{J_s(k) \mu(k/(j, k^s)_s)}{J_s(k/(j, k^s)_s)}$$

Now we group the terms in the sum according to $d^s = (j, k^s)_s$,

$$\begin{aligned} \sum_{j=1}^{k^s} f((j, k^s)_s) c_k^{(s)}(j) &= J_s(k) \sum_{d^s | k^s} f(d^s) \frac{\mu(\frac{k^s}{d^s})}{J_s(\frac{k^s}{d^s})} \sum_{\substack{m=1 \\ (m, \frac{k^s}{d^s})_s=1}}^{\frac{k^s}{d^s}} 1 \\ &= J_s(k) \sum_{d^s | k^s} f(d^s) \frac{\mu(\frac{k^s}{d^s})}{J_s(\frac{k^s}{d^s})} J_s(\frac{k^s}{d^s}) \\ &= J_s(k) \sum_{d^s | k^s} f(d^s) \mu(\frac{k^s}{d^s}) \end{aligned}$$

which is equivalent to the identity we have to prove. \square

7. THE GAMMA FUNCTION AS WEIGHT

To establish the next average, we need a few preliminary lemmas. We have the following identity which appears in [9] as exercise 1.8.45:

Lemma 14. For $s, k \in \mathbb{N}$

$$\sum_{d|k} \frac{\mu(d)}{d} \log d = -\frac{\varphi(k)}{k} \sum_{\substack{p|k \\ p \text{ prime}}} \frac{\log p}{p-1}$$

We will give a generalization of this lemma in terms of Jordan totient function. This lemma could be of some other independent interest.

Lemma 15. For $s, k \in \mathbb{N}$

$$\sum_{d|k} \frac{\mu(d)}{d^s} \log d = -\frac{J_s(k)}{k^s} \sum_{\substack{p|k \\ p \text{ prime}}} \frac{\log p}{p^s - 1}$$

Proof. When we take all the divisors of k , only those $\mu(d)$ survive in the computation where d is square free. Now recall that

$$\frac{J_s(k)}{k^s} = \prod_{\substack{p|k \\ p \text{ prime}}} \left(1 - \frac{1}{p^s}\right).$$

So the quotient $\frac{J_s(k)}{k^s}$ considers each prime factor in a divisor d of k only once. Therefore, without any loss of generality we may assume that k is square free. Let $k = \prod_{p_i|k} p_i$ where p_i are distinct primes and $D = \{p_1, \dots, p_r\}$ denote

the set of all prime divisors of k . With this assumption and notation, the RHS of the expression we want to verify becomes

$$\begin{aligned}
\frac{J_s(k)}{k^s} \sum_{p_i \in D} \frac{\log p_i}{p_i^s - 1} &= \prod_{p_i \in D} \left(1 - \frac{1}{p_i^s}\right) \sum_{p_i \in D} \frac{\log p_i}{p_i^s - 1} \\
&= \prod_{p_i \in D} \frac{p_i^s - 1}{p_i^s} \frac{\sum_{p_i \in D} \left(\prod_{\substack{p_j \in D \\ p_j \neq p_i}} (p_j^s - 1) \log p_i \right)}{\prod_{p_i \in D} (p_i^s - 1)} \\
&= \frac{\sum_{p_i \in D} \left(\prod_{\substack{p_j \in D \\ p_j \neq p_i}} (p_j^s - 1) \log p_i \right)}{\prod_{p_i \in D} p_i^s}
\end{aligned}$$

We introduce a notation for convenience. S_{n,p_i} will stand for the set of n primes (where $n < r$) from D and contains p_i and S_n will stand for the set of all S_{n,p_i} where $p_i \in D$. Therefore, $S_{r,p_i} = D$. With this notation, we get

$$\begin{aligned}
\frac{J_s(k)}{k^s} \sum_{p_i \in D} \frac{\log p_i}{p_i^s - 1} &= \left(\frac{1}{\prod_{p_i \in D} p_i^s} \right) \left[\sum_{p_i \in D} \log p_i \left(\prod_{\substack{p_j \in D \\ p_j \neq p_i}} (p_j^s - 1) \right) \right] \\
&= \left(\frac{1}{\prod_{p_i \in D} p_i^s} \right) \left[\sum_{p_i \in D} \log p_i \left(\prod_{\substack{p_j \in D \\ p_j \neq p_i}} p_j^s - \sum_{S_{r-1,p_i} \in S_{r-1}} \prod_{\substack{p_j \in S_{r-1} \\ p_j \neq p_i}} p_j^s + \dots + \right) \right] \\
&= \sum_{p_i \in D} \log p_i \left(\frac{1}{p_i^s} - \sum_{S_{2,p_i} \in S_2} \frac{1}{\prod_{p_j \in S_2} p_j^s} + \dots + \right)
\end{aligned}$$

Computing the LHS now:

$$\begin{aligned}
\sum_{d|k} \frac{\mu(d)}{d^s} \log d &= -\frac{\log p_1}{p_1^s} \\
&+ -\left(\frac{\log p_1 p_2}{p_1^s p_2^s} + \dots + \frac{\log p_{r-1} p_r}{p_{r-1}^s p_r^s} \right) \\
&+ \frac{\log p_1 p_2 p_3}{p_1^s p_2^s p_3^s} + \dots + \frac{\log p_{r-2} p_{r-1} p_r}{p_{r-2}^s p_{r-1}^s p_r^s} \\
&+ \dots + (-1)^r \frac{\log p_1 \dots p_r}{p_1^s \dots p_r^s}
\end{aligned}$$

grouping the terms on p_i ,

$$\begin{aligned}
\sum_{d|k} \frac{\mu(d)}{d^s} \log d &= \sum_{p_i \in D} \log p_i \left(-\frac{1}{p_i^s} + \sum_{S_{2,p_i} \in S_2} \frac{1}{\prod_{p_j \in S_{2,p_i}} p_j^s} - \sum_{S_{3,p_i} \in S_3} \frac{1}{\prod_{p_j \in S_{3,p_i}} p_j^s} \right. \\
&+ \dots \\
&+ (-1)^k \sum_{S_{k,p_i} \in S_k} \frac{1}{\prod_{p_j \in S_{k,p_i}} p_j^s} \\
&+ \dots + (-1)^{r-1} \sum_{S_{r-1,p_i} \in S_{r-1}} \frac{1}{\prod_{p_j \in S_{r-1,p_i}} p_j^s} \\
&\left. + (-1)^r \frac{1}{\prod_{p_j \in D} p_j^s} \right)
\end{aligned}$$

which is the same as the RHS computed above. \square

The above lemma will be used in the proof of the next proposition.

Proposition 16. *For $s, k \in \mathbb{N}$*

$$\frac{1}{J_s(k)} \sum_{j=1}^{k^s} \log \Gamma(j/k^s) c_k^{(s)}(j) = \frac{s}{2} \sum_{p|k} \frac{\log p}{p^s - 1} - \frac{\log 2\pi}{2}$$

Proof.

$$\begin{aligned}
\sum_{j=1}^{k^s} \log \Gamma(j/k^s) c_k^{(s)}(j) &= \sum_{j=1}^{k^s} \log \Gamma(j/k^s) \sum_{d^s | j, d^s | k^s} d^s \mu\left(\frac{k}{d}\right) \\
&= \sum_{d^s | k^s} d^s \mu\left(\frac{k}{d}\right) \sum_{m=1}^{\frac{k^s}{d^s}} \log \Gamma(md^s/k^s) \\
&= \sum_{d^s | k^s} d^s \mu\left(\frac{k}{d}\right) \log \prod_{m=1}^{\frac{k^s}{d^s}} \Gamma\left(\frac{m}{\frac{k^s}{d^s}}\right)
\end{aligned}$$

Now [9], Proposition 9.6.33 tells that

$$\prod_{1 \leq j \leq N} \Gamma(j/N) = \frac{(2\pi)^{(n-1)/2}}{\sqrt{n}}$$

Using this, we get

$$\begin{aligned}
\sum_{j=1}^{k^s} \log \Gamma(j/k^s) c_k^{(s)}(j) &= \sum_{d^s | k^s} d^s \mu\left(\frac{k}{d}\right) \log \frac{(2\pi)^{(k^s-1)/2}}{\sqrt{\frac{k^s}{d^s}}} \\
&= \sum_{d^s | k^s} d^s \mu\left(\frac{k}{d}\right) \left(\frac{1}{2} \left(\frac{k^s}{d^s} - 1 \right) \log 2\pi - \frac{1}{2} \log \frac{k^s}{d^s} \right) \\
&= \frac{k^s}{2} \log 2\pi \sum_{d|k} \mu\left(\frac{k}{d}\right) - \frac{1}{2} \log 2\pi \sum_{d|k} d^s \mu\left(\frac{k}{d}\right) - \frac{1}{2} \sum_{d|k} d^s \mu\left(\frac{k}{d}\right) \log\left(\frac{k^s}{d^s}\right)
\end{aligned}$$

It is a standard fact that (see [4]) $\sum_{d|k} \mu\left(\frac{k}{d}\right) = 0$. Also, taking k/d instead of d as the divisors of k , we get

$$\begin{aligned}
\sum_{j=1}^{k^s} \log \Gamma(j/k^s) c_k^{(s)}(j) &= -\frac{1}{2} J_s(k) \log 2\pi - \frac{1}{2} \sum_{d|k} \frac{k^s}{d^s} \mu(d) \log(d^s) \\
&= -\frac{1}{2} J_s(k) \log 2\pi - \frac{sk^s}{2} \times -\frac{J_s(k)}{k^s} \sum_{\substack{p|k \\ p \text{ prime}}} \frac{\log p}{p^s - 1} \\
&\quad \text{using Lemma (15)}
\end{aligned}$$

$$= J_s(k) \left(\frac{s}{2} \sum_{\substack{p|k \\ p \text{ prime}}} \frac{\log p}{p^s - 1} - \frac{1}{2} \log 2\pi \right)$$

and this completes the proof. \square

8. BERNOULLI NUMBERS AS WEIGHT

We now consider the weighted average involving Bernoulli numbers. Even though a similar identity was stated in [18], no proof was given there. Our next proposition gives a generalization of this proposition, and a special case to of turns out to be Tóth's identity given in [18].

Proposition 17. *For $m \in \mathbb{Z}, m \geq 0$*

$$\frac{1}{k^s} \sum_{j=0}^{k^s-1} B_m\left(\frac{j}{k^s}\right) c_k^{(s)}(j) = \frac{B_m}{k^{sm}} J_{sm}(k)$$

Proof.

$$\begin{aligned}
\sum_{j=0}^{k^s-1} B_m\left(\frac{j}{k^s}\right) c_k^{(s)}(j) &= \sum_{j=0}^{k^s-1} B_m\left(\frac{j}{k^s}\right) d^s \mu\left(\frac{k}{d}\right) \\
&= \sum_{d|k} d^s \mu\left(\frac{k}{d}\right) \sum_{\substack{j=0 \\ d^s | j}}^{k-1} B_m\left(\frac{j}{k}\right)
\end{aligned}$$

Now use the fact that $B_m(0) = B_m(1)$.

$$\begin{aligned}
\sum_{j=0}^{k^s-1} B_m\left(\frac{j}{k^s}\right) c_k^{(s)}(j) &= \sum_{d|k} d^s \mu\left(\frac{k}{d}\right) \left(\sum_{\substack{j=1 \\ d^s|j}}^{k^s} B_m\left(\frac{j}{k^s}\right) + B_m\left(\frac{0}{k^s}\right) - B_m\left(\frac{k^s}{k^s}\right) \right) \\
&= \sum_{d|k} d^s \mu\left(\frac{k}{d}\right) \sum_{\substack{j=1 \\ d^s|j}}^{k^s} B_m\left(\frac{j}{k^s}\right) \\
&= \sum_{d|k} d^s \mu\left(\frac{k}{d}\right) \sum_{r=1}^{\frac{k^s}{d^s}} B_m\left(\frac{rd^s}{k^s}\right) \\
&= \sum_{d|k} d^s \mu\left(\frac{k}{d}\right) \sum_{r=1}^{\frac{k^s}{d^s}} B_m\left(\frac{r}{\frac{k^s}{d^s}}\right) \\
&= \sum_{d|k} d^s \mu\left(\frac{k}{d}\right) \sum_{r=0}^{\frac{k^s}{d^s}-1} B_m\left(\frac{r}{\frac{k^s}{d^s}}\right) - B_m\left(\frac{0}{\frac{k^s}{d^s}}\right) + B_m\left(\frac{\frac{k^s}{d^s}}{\frac{k^s}{d^s}}\right)
\end{aligned}$$

From the properties Bernoulli numbers ([9], Prop. 9.1.3), we have

$$\sum_{j=0}^{k-1} B_m\left(\frac{j}{k}\right) = \frac{B_m}{k^{m-1}}$$

Using this, we get

$$\begin{aligned}
\sum_{j=0}^{k^s-1} B_m\left(\frac{j}{k^s}\right) c_k^{(s)}(j) &= \sum_{d|k} d^s \mu\left(\frac{k}{d}\right) \frac{B_m}{\left(\frac{k^s}{d^s}\right)^{m-1}} \\
&= \frac{B_m}{(k^s)^{m-1}} \sum_{d|k} d^{sm} \mu\left(\frac{k}{d}\right) \\
&= \frac{B_m}{(k^s)^{m-1}} J_{sm}(k)
\end{aligned}$$

Now the conclusion easily follows. □

9. BINOMIAL COEFFICIENTS AS WEIGHT

Now we have one more weighted average with weights as binomial coefficients.

Proposition 18. *For $k \in \mathbb{N}$,*

$$\sum_{j=0}^{k^s} \binom{k^s}{j} c_k^{(s)}(j) = 2^{k^s} \sum_{d|k} \mu\left(\frac{k}{d}\right) \sum_{l=1}^{d^s} (-1)^{l \frac{k^s}{d^s}} \cos^{k^s}(l\pi/d^s)$$

Proof.

$$\begin{aligned}
\sum_{j=0}^{k^s} \binom{k^s}{j} c_k^{(s)}(j) &= \sum_{j=0}^{k^s} \binom{k^s}{j} \sum_{d|k, d^s|l} d^s \mu\left(\frac{k}{d}\right) \\
&= \sum_{d|k} d^s \mu\left(\frac{k}{d}\right) \sum_{m=0}^{\frac{k^s}{d^s}} \binom{k^s}{md^s} \\
&= \sum_{d|k} d^s \mu\left(\frac{k}{d}\right) \sum_{m=0}^{\frac{k^s}{d^s}} \binom{k^s}{md^s}
\end{aligned}$$

Now we use the following formula appearing in [10], P. 84:

$$\sum_{m=0}^{\lfloor n/r \rfloor} \binom{n}{mr} = \frac{2^n}{r} \sum_{l=1}^r \cos^n(l\pi/r) \cos(nl\pi/r)$$

to get

$$\sum_{j=0}^{k^s} \binom{k^s}{j} c_k^{(s)}(j) = \sum_{d|k} d^s \mu\left(\frac{k}{d}\right) \frac{2^{k^s}}{d^s} \sum_{l=1}^{d^s} \cos^{k^s}(l\pi/d^s) \cos(k^s l\pi/d^s)$$

Note that k^s/d^s is an integer and so $\cos(k^s l\pi/d^s)$ is $+1$ if $\frac{k^s}{d^s}l$ is even and -1 if it is odd. So we get the final expression which was claimed. \square

10. EXPONENTIAL FUNCTION AS WEIGHT

Though the following proposition was stated in [18] as a remark, detailed proof was not given there. We provide a proof here before generalizing the proposition.

Proposition 19. *For every $k, n \in \mathbb{N}$,*

$$\frac{1}{k} \sum_{j=1}^k \exp(2\pi i j n / k) c_k(j) = \begin{cases} 1, & \text{if } (k, n) = 1 \\ 0, & \text{if } (k, n) > 1 \end{cases}$$

Proof. We use the Fourier expansion of the functions $\theta_k(n)$ (defined in the section *Notations*) and $c_k(n)$ to prove the identity. We have

$$\frac{1}{k} \sum_{j=1}^k \theta_k(j) \exp(-2\pi i j n / k) = \frac{1}{k} \sum_{\substack{j=1 \\ (j,k)=1}}^k \exp(-2\pi i j n / k) = \frac{c_k(n)}{k}.$$

Therefore Using the Fourier expansion identities, we get

$$\sum_{j=1}^k \frac{c_k(j)}{k} \exp(2\pi i j n / k) = \theta_k(n)$$

\square

The generalization follows easily:

Proposition 20. *For $k, n \in \mathbb{N}$, we have*

$$\frac{1}{k^s} \exp(2\pi i j n / k^s) c_k^{(s)}(j) = \begin{cases} 1 & \text{if } (k, n^s)_s = 1 \\ 0 & \text{if } (k, n^s)_s > 1 \end{cases} =: \theta_k^{(s)}(n)$$

Proof.

$$\frac{1}{k^s} \sum_{j=1}^{k^s} \theta_k^{(s)}(j) \exp(-2\pi i j n / k^s) = \frac{1}{k^s} \sum_{\substack{j=1 \\ (j, k^s)_s = 1}}^{k^s} \exp(-2\pi i j n / k^s) = \frac{c_k^{(s)}(n)}{k^s}.$$

Threfore Using the Fourier expansion identities, we get

$$\sum_{j=1}^{k^s} \frac{c_k^{(s)}(j)}{k^s} \exp(2\pi i j n / k^s) = \theta_k^{(s)}(n)$$

which is what we require. \square

11. GENERALIZATION OF A MULTIVARIATE IDENTITY

Finally we have a generalization of a multivariate identity given in [18]:

Proposition 21. *Let $k_1, \dots, k_n \in \mathbb{N}$, $k = \text{lcm}(k_1, \dots, k_n)$ and let $s, r \in \mathbb{N}$. Then*

$$(7) \quad S_r^{(s)}(k_1, \dots, k_n) = \frac{J_s(k_1) \dots J_s(k_n)}{2k^s} + \frac{1}{r+1} \sum_{m=0}^{\lfloor r/2 \rfloor} \binom{r+1}{2m} \frac{B_{2m}}{k^{2ms}} g_m(k_1, \dots, k_n)$$

where

$$(8) \quad g_m^{(s)}(k_1, \dots, k_n) = \sum_{d_1 | k_1, \dots, d_n | k_n} \frac{d_1^s \mu(k_1/d_1) \dots d_n^s \mu(k_n/d_n)}{(\text{lcm}(d_1, \dots, d_n))^{(1-2m)s}}$$

is a multiplicative function in n variables.

This can be easily verified following the steps given in the proof of the single variable case. With $r = 1$, we have

Corollary 22. *With notations as in the above proposition, if we let $E^{(s)} := g_1$, then*

$$S_1^{(s)}(k_1, \dots, k_n) = \frac{J_s(k_1) \dots J_s(k_n)}{2k} + \frac{E^{(s)}(k_1, \dots, k_n)}{2}$$

12. FURTHER DIRECTIONS

Though we have found several identities for various arithmetical functions combined with the generalized Ramanujan sums, we restricted ourselves to the finite summation case. If one looks into what happens when we extend this to an infinite series, it is possible that it may give us many more ‘‘Ramanujan expansions’’ of many well known arithmetical functions in terms of the generalized sums. Also, it should be possible to derive similar identities based on the other generalizations of Ramanujan sums with the same techniques.

REFERENCES

- [1] E. Alkan, On the mean square average of special values of L -functions, *Journal of Number Theory* **131** (2011), 1470-1485
- [2] E. Alkan, Distribution of averages of Ramanujan sums, *Ramanujan Journal* **29** (2012), 385-408.
- [3] E. Alkan, Ramanujan sums and the Burgess zeta function, *Int. J. Number Theory* **8** (2012), 2069-2092.
- [4] T. M. Apostol, *Introduction to Analytic Number Theory*, Undergraduate Texts in Mathematics, Springer, New York (1976)
- [5] J. Chidambaraswamy, Generalized Ramanujan's Sum, *Periodica Math. Hungarica*, **10** (1979), 71-87
- [6] H. Cohen, An extension of Ramanujan's Sum, *Duke Math. J.* **16** (1949), 85-90
- [7] H. Cohen, An extension of Ramanujan's Sum II. Additive Properties, *Duke Math. J.* **22** (1955), 543 - 559
- [8] H. Cohen, An extension of Ramanujan's Sum III. Connections with Totient Functions, *Duke Math. J.* **23** (1956), 623 - 630.
- [9] H. Cohen, *Number Theory Vol. II. Analytic and modern tools*, Graduate Texts in Mathematics, 240, Springer, (2007).
- [10] L. Comtet, *Advanced Combinatorics*, Reidel Publishing Co, The Art of Finite and Infinite Expansions. D (1974)
- [11] Hölder, Zur Theorie der Kreisteilungsgleichung $K_m(x) = 0$, *Prace Matematyczno Fizyczne*, **43** (1936), 13-23.
- [12] Gordon James, Martin Liebeck. *Representations and characters of groups*, Cambridge University Press, New York, second edition, (2001)
- [13] Lutz G. Lucht, A survey of Ramanujan expansions. *Int. J. Number Theory*, **6(8)** (2010) 1785-1799
- [14] Melvyn B. Nathanson. *Additive number theory*, volume 164 of Graduate Texts in Mathematics. Springer-Verlag, New York (1996). The classical bases.
- [15] S. Ramanujan, On certain trigonometrical sums and their applications in the theory of numbers, *Trans. Cambridge Philos. Soc.* **22** (1918), 259-276 (Collected Papers, Cambridge 1927, No. 21).
- [16] W. Schwarz, J. Spilker, *Arithmetical Functions*, London Mathematical Society Lecture Note Series, 184, Cambridge University Press, (1994).
- [17] J. Singh, Defining power sums of n and $\varphi(n)$ integers, *Int. J. Number Theory* **5** (2009), 41-53.
- [18] L. Tóth, Averages of Ramanujan sum: Note on two papers by E. Alkan, *Ramanujan J.* **35** (2014) 149 - 156
- [19] C.S. Venkataraman and R. Sivaramakrishnan, An extension of Ramanujan's sum, *Math. Student* **40A** (1972), 211-216